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MATHEMATICAL TREATMENT OF THE DISCHARGE OF A LAMINAR HOT GAS IN A STAGNANT COLDER ATMOSPHERE

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We study the boundary-layer approximation of the classical mathematical model that describes the discharge of a laminar hot gas in a stagnant colder atmosphere of the same gas. We prove the existence and uniqueness of solutions to a nondegenerate problem (without zones of stagnation of gas temperature or velocity). The asymptotic behavior of these solutions is also studied

Key words: *systems of nonlinear degenerate parabolic equations, diffusion coupling, temperature gas jets, asymptotic behavior.*

Introduction. In this paper, we consider a mathematical model describing the discharge of a laminar hot gas in a stagnant colder atmosphere of the same gas. The problem is already classical in the theory of compressible fluid dynamics, and it has an important relevance in combustion. We consider a boundary-layer approximation (which allows the pressure effects to be neglected). Dimensionless modelling of planar jets, according to the boundary-layer approach, yields a nonlinear system of partial differential equations (see [1–7])

$$\frac{\partial\,(r^i\rho u)}{\partial x}+\frac{\partial\,(r^i\rho v)}{\partial r}=0,$$
$$r^i\rho u\,\frac{\partial u}{\partial x}+r^i\rho v\,\frac{\partial u}{\partial r}=\frac{\partial}{\partial r}\Big(r^i\mu\,\frac{\partial u}{\partial r}\Big)+G\Big(1-\frac{\varepsilon}{T}\Big), \tag{1}$$
$$r^i\rho u\,\frac{\partial T}{\partial x}+r^i\rho v\,\frac{\partial T}{\partial r}=\frac{1}{\mathrm{Pr}}\,\frac{\partial}{\partial r}\Big(r^i\mu\,\frac{\partial T}{\partial r}\Big),$$

where Pr is the Prandtl number and G is the inverse squared Froude number (positive numbers are given). The superscript i takes the value $i=0$ for the planar configuration and $i=1$ for the axisymmetric jet. The system is closed with the constitutive conditions $\rho=1/T$ and $\mu=T^\sigma$ $(0<\sigma<\infty)$. Here the unknowns are the velocity components v and u and the temperature T.

System (1) is considered in the domain $\Omega=\{(x,r)\in\mathbb{R}^2\colon\ 0<x<\infty,\ 0<r<l\le\infty\}$ with the boundary conditions

$$\frac{\partial u}{\partial r}=v=\frac{\partial T}{\partial r}=0,\qquad r=0,\quad x>0,$$
$$u=\delta,\quad T=\varepsilon,\qquad r=l,\quad x>0 \tag{2}$$

and the "initial" condition

$$u(0,r)=u_0(r)\ge\delta,\qquad T(0,r)=T_0(r)\ge\varepsilon,\qquad x=0,\quad r\in[0,l]. \tag{3}$$

Notice that, although arising in the stationary regime, the system is of parabolic type and that condition (6) looks as an initial condition if we understand the variable x as a fictitious time. Problem (1)–(3) was considered in [7] for the case

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$$T \to \varepsilon, \quad u \to \delta \quad \text{as} \quad r \to +\infty.$$

In the same paper, the authors also considered self-similar solutions and their numerical simulations.

Here we prove the existence and uniqueness of solutions of the nondegenerate problem (corresponding to the assumptions $\delta > 0$ and $\varepsilon > 0$, which means that possible zones of stagnation of gas temperature or velocity are absent). We also study the asymptotic behavior of solutions with respect to x and l.

To prove the existence and uniqueness of solutions, we use the so-called von Mises variables x and ψ (ψ is the associated "stream function"), which transform system (1) into a purely diffusive system (after elimination of the unknown v):

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial \psi}\Big(r^{2i} T^{\sigma-1} u \, \frac{\partial u}{\partial \psi}\Big) + \frac{TG}{ur^i}\Big(1 - \frac{\varepsilon}{T}\Big), \qquad \frac{\partial T}{\partial x} = \frac{1}{\Pr}\, \frac{\partial}{\partial \psi}\Big(r^{i} T^{\sigma-1} u \, \frac{\partial T}{\partial \psi}\Big).$$

Notice that systems of this nature arise in many different contexts, such as biology or filtration problems.

1. Von Mises Transformation to the Variables *x* and *ψ*. We introduce the stream function $\psi(x, r)$ by the formulas

$$\psi_r = r^i \rho u, \qquad \psi_x = -r^i \rho v,$$

and then we define new independent (von Mises) variables

$$(X = x, \ \psi = \psi(x,r)) \leftrightarrow (x, r), \qquad \frac{D(X, \psi)}{D(x, r)} = \psi_r = r^i \rho u > 0, \tag{4}$$

which determine the homeomorphism $(x, r) \leftrightarrow (X = x, \ \psi = \psi(x, r))$. For simplicity in the notation of the further study, we conserve the notation $X = x$. Using the formulas

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial X} - r^i \rho v \, \frac{\partial}{\partial \psi}, \qquad \frac{\partial}{\partial r} = r^i \rho u \, \frac{\partial}{\partial \psi}$$

and eliminating the unknown v, we transform system (1) to a system of equations for the unknowns u and T

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial \psi}\Big(r^{2i} a(u,T)\, \frac{\partial u}{\partial \psi}\Big) + r^{-i} c(u,T), \qquad \frac{\partial T}{\partial x} = \frac{\partial}{\partial \psi}\Big(r^{2i} b(u,T)\, \frac{\partial T}{\partial \psi}\Big), \tag{5}$$

where

$$a = T^{\sigma-1} u, \qquad c = \frac{G}{u} T\Big(1 - \frac{\varepsilon}{T}\Big), \qquad b = \frac{a}{\Pr}. \tag{6}$$

Here $r = r(x, \psi)$ is defined by means of a nonlocal operator

$$r^{i+1}(x, \psi) = (i+1) \int_0^{\psi} \frac{T(x,s)}{u(x,s)}\, ds.$$

As was mentioned before, systems similar to (5) appear also in other different contexts, such as biology or filtration problems (see, e.g. [8–11]).

Note, for $\sigma = 1$, $i = 0$, and $G = 0$, system (5) splits into two independent parabolic equations. In this case, the first equation coincides with the famous "porous medium equation" for the function u (see, e.g. [12–17]). The second equation can be understood as a "generalized porous medium equation" for the function T with a given function u.

2. Existence and Uniqueness of Solutions for Nondegenerate Problems ($0 < \varepsilon \le 1$ and $0 < \delta \le 1$). In the planar case ($i = 0$), we consider two different problems associated with system (5).

Problem 1. Given $0 < l < \infty$, find a solution (u, T) of system (5) in the domain $\Omega = \{(x, \psi) \in \mathbb{R}^2\colon 0 < x < \infty,\ 0 < \psi < l < \infty\}$ with the boundary conditions

$$\frac{\partial u}{\partial \psi} = \frac{\partial T}{\partial \psi} = 0 \qquad \text{for} \quad \psi = 0, \quad x > 0,$$

$$u = \delta, \qquad T = \varepsilon \qquad \text{for} \quad \psi = l, \quad x > 0$$

and the "initial" conditions

$$u = u_0(\psi), \quad T = T_0(\psi) \quad \text{for} \quad x = 0, \quad 0 \le \psi \le l. \tag{7}$$

System (5) is of parabolic type; hence, condition (7) can be interpreted as an initial condition if the variable x is considered as a "fictitious" time. In Eqs. (7), $u_0(\psi) = u_0(r(0,\psi))$ and $T_0(\psi) = T_0(r(0,\psi))$ with the function $r(0,\psi)$ defined by

$$r(0,\psi) = \int_0^{\psi} \frac{T_0(s)}{u_0(s)}\, ds.$$

We assume that

$$u_0(l) = \delta, \quad T_0(l) = \varepsilon, \quad 0 < \delta \le u_0(\psi) \le 1, \quad 0 < \varepsilon \le T_0(\psi) \le 1. \tag{8}$$

Problem 2. Find a solution (u,T) of system (5) in the domain $\Omega = \{(x,\psi) \in \mathbb{R}^2\colon\ 0 < x < \infty,\ 0 < \psi < f(x) = \psi(x,l)\}$ with $r(x,f(x)) = l = \text{const}$ under the boundary and "initial" conditions

$$\frac{\partial u}{\partial \psi} = \frac{\partial T}{\partial \psi} = 0 \qquad \text{for} \quad \psi = 0, \quad x > 0,$$

$$u = \delta, \qquad T = \varepsilon \qquad \text{for} \quad \psi = \psi(x,l) = f(x), \quad x > 0, \tag{9}$$

$$u = u_0(\psi), \qquad T = T_0(\psi) \qquad \text{for} \quad x = 0, \quad 0 \le \psi < l.$$

Here $\psi(x,l) = f(x)$ is an unknown function defined by the equation

$$r(x,\psi) = l, \qquad 0 < l < \infty$$

with a given positive constant l. Notice that Problem 1 and Problem 2 coincide if $l = \infty$.

2.1. *Existence and Uniqueness of Solutions of Problem* 1. We start by proving the following "maximum principle" for both problems. We assume that $G = 0$; then, we have

$$\delta \le u(x,\psi) \le 1, \qquad \varepsilon \le T(x,\psi) \le 1. \tag{10}$$

Moreover, if $G > 0$, then, for $x \in [0,X]$, we obtain

$$\delta \le u(x,\psi) \le \inf_{\lambda>0} [\max(\mathrm{e}^{\lambda X}\sqrt{G}/\sqrt{\lambda}, 1)] = C_0(X), \qquad \varepsilon \le T(x,\psi) \le 1. \tag{11}$$

Under conditions (8) and (9), this statement follows from the classical maximum principle (see, e.g., [18, Chapter 1]). Estimates (10) and (11) lead to the inequalities

$$0 < a_0(\varepsilon,\delta) \le a = b\mathrm{Pr} \le a_1(\varepsilon,\delta) < \infty, \qquad 0 \le c \le c_0(\varepsilon,\delta) < \infty. \tag{12}$$

Theorem 2.1. *Let $(u_0,T_0) \in C^{\alpha}[0,l]$, $0 < \alpha < 1$, satisfying conditions* (8) *with $0 < \varepsilon$, $0 < \delta$. Then, for any given $0 < X < \infty$, Problem* 1 *has at least the classical solution $(u,T) \in C^{\alpha,\alpha/2}(\overline{\Omega}) \cap C^{2m+\alpha,m+\alpha/2}(\Omega')$ $[\Omega = (0,X)\times(0,l)$, $\Omega' = \Omega \cap (x > 0)$, and $m \ge 1]$. Moreover, the solution is unique if $(u_\psi, T_\psi) \in L^4(0,X; L^{2q}(0,l))$ for some $q > 1$.*

Proof. According to (12), system (5) is uniformly parabolic with regular bounded coefficients if $\varepsilon > 0$ and $\delta > 0$. Then, by applying some well-known results (see, e.g., [18, Chapter 5]), we prove the existence of the solution.

To prove the uniqueness of the solution, we use the usual way. Let $\boldsymbol{w}_i = (u_i, T_i)$ $(i = 1,2)$ be two different solutions and

$$\boldsymbol{w} = (u,T) = (u_1 - u_2, T_1 - T_2).$$

Then, we can easily see that u and T satisfy the system of equations

$$\frac{\partial \boldsymbol{w}}{\partial x} = \frac{\partial}{\partial \psi}\Big(A\,\frac{\partial \boldsymbol{w}}{\partial \psi} + B\boldsymbol{w}\Big) + C\boldsymbol{w}, \tag{13}$$

where the matrices A, B, and C are defined by

$$A = \{A_{ij}\}, \qquad A_{11} = a_{11}, \qquad A_{22} = b_{11}, \qquad A_{12} = A_{21} = 0,$$

$$B = \{B_{ij}\}, \qquad B_{11} = \frac{a_{11} - a_{21}}{u_1 - u_2}\, u_{2\psi}, \qquad B_{12} = \frac{a_{21} - a_{22}}{T_1 - T_2}\, u_{2\psi},$$

$$B_{21} = \frac{b_{11} - b_{21}}{u_1 - u_2} T_{2\psi}, \qquad B_{22} = \frac{b_{21} - b_{22}}{T_1 - T_2} T_{2\psi},$$

$$C = \{C_{ij}\}, \qquad C_{11} = \frac{c_{11} - c_{21}}{u_1 - u_2}, \qquad C_{12} = \frac{c_{21} - c_{22}}{T_1 - T_2}, \qquad C_{21} = C_{22} = 0,$$

$$a_{ij} = a(u_i, T_j), \qquad b_{ij} = b(u_i, T_j), \qquad c_{ij} = c(u_i, T_j).$$

It is easy to verify that

$$0 < a_0(\varepsilon, \delta) \le A_{11}, \qquad A_{22} \le b_0(\varepsilon, \delta),$$

$$|B| \le K|\boldsymbol{w}_{2\psi}|, \qquad |C_i| \le K, \qquad K = \max(|a_{\boldsymbol{w}}|, |b_{\boldsymbol{w}}|, |c_{\boldsymbol{w}}|).$$

Multiplying (13) by $\boldsymbol{w}$ and integrating over $(0,x) \times (0,l)$, we come to the inequality

$$\int_0^l |\boldsymbol{w}(x,s)|^2 \, ds + \int_0^x \int_0^l |\boldsymbol{w}_\psi|^2 \, ds \, dt \le CI, \qquad x > 0, \tag{14}$$

where

$$I = \int_0^x \int_0^l |\boldsymbol{w}(x,s)|^2 |\boldsymbol{w}_{2\psi}|^2 \, ds \, dt, \qquad C = C(K)$$

and

$$I \le \int_0^x \Big(\int_0^l |\boldsymbol{w}(x,s)|^{2q/(q-1)} \, ds \Big)^{(q-1)/q} \Big(\int_0^l |\boldsymbol{w}_{2\psi}|^{2q} \, ds \Big)^{1/q} dt, \qquad 1 < q < \infty.$$

Next, we introduce the function

$$Y(x) = \int_0^l |\boldsymbol{w}(x,s)|^2 \, ds$$

and use the inequalities

$$|\boldsymbol{w}(x,s)|^2 \le 4Y^{1/2}(x) \Big(\int_0^l |\boldsymbol{w}_\psi|^2 \, ds \Big)^{1/2},$$

$$\Big(\int_0^l |\boldsymbol{w}(x,s)|^{2q/(q-1)} \, ds \Big)^{(q-1)/q} \le 4 l^{(q-1)/q} Y^{1/2}(x) \Big(\int_0^l |\boldsymbol{w}_\psi|^2 \, ds \Big)^{1/2},$$

$$I \le \int_0^x 4 l^{(q-1)/q} Y^{1/2}(x) \Big(\int_0^l |\boldsymbol{w}_\psi|^2 \, ds \Big)^{1/2} \Big(\int_0^l |\boldsymbol{w}_{2\psi}|^{2q} \, ds \Big)^{1/q} dt$$

$$\le \frac{C(K)}{2} \int_0^x \int_0^l |\boldsymbol{w}_\psi|^{2q} \, ds \, dt + C(K,l,q) \int_0^x Y(t) \Big(\int_0^l |\boldsymbol{w}_{2\psi}|^{2q} \, ds \Big)^{2/q} dt.$$

Joining (14) and the last inequalities, we come to the integral inequality

$$Y(x) \le C \int_0^x Y(s) \Big(\int_0^l |\boldsymbol{w}_{2\psi}|^{2q} \, dt \Big)^{2/q} ds,$$

which has only the trivial solution if

$$\int_0^x \Big(\int_0^l |\boldsymbol{w}_{2\psi}|^{2q}\, dt\Big)^{2/q} ds \le C < \infty, \qquad q > 1. \tag{15}$$

This means that any solution satisfying inequality (15) is unique.

Remark 1. The constructed solution $\boldsymbol{w} = (u(x,\psi), T(x,\psi))$ defines a homeomorphism between the domain in the plane of the physical variables $\Omega_{X,r} = \{(x,\psi) \in \mathbb{R}^2\colon\ 0 < x < X,\ 0 < r < r(x,l)\}$ and the domain $\Omega_{X,l} = \{(x,\psi) \in \mathbb{R}^2\colon\ 0 < x < X,\ 0 < \psi < l\}$ with the Jacobian defined by

$$0 < \varepsilon \le \frac{D(X,\psi)}{D(x,r)} = \psi_r = \rho u = \frac{T}{u} \le \frac{1}{\delta} < \infty.$$

This solution determines also the form of the streamline $r = r(x,l) = g(x)$ $[g'(x) = v(x,l)/u(x,l)]$ in the physical domain. The second component of the velocity vector $v(x,\psi)$ is defined by the formula

$$v(x,\psi) = u(x,\psi)\frac{\partial}{\partial x}\int_0^\psi \frac{ds}{\rho(x,s)u(x,s)}. \tag{16}$$

According to Eqs. (4), (10), and (11), the classical solution $\boldsymbol{w} = (u(x,\psi), T(x,\psi))$ determines the classical solution $(v(x,r), u(x,r), T(x,r))$ of system (1) satisfying conditions (2).

Remark 2. The existence of solutions for the axisymmetric case ($i = 1$) can be proved in a similar way, but with some slight technical complications.

Let us consider the asymptotic behavior of solutions of Problem 1 with respect to the variable x. We introduce the energy functions

$$\eta_l(x) = \int_0^l (u(x,\psi) - \delta)^2\, d\psi, \qquad \mu_l(x) = \int_0^l (T(x,\psi) - \varepsilon)^2\, d\psi$$

and the positive numbers [see (6)]

$$\lambda = a_0/(2l^2 \mathrm{Pr}\,), \qquad \nu = a_0/(2l^2) \quad (\nu = \lambda \mathrm{Pr}\,).$$

First we consider the case with $0 < l < \infty$.

Theorem 2.2. *Let $\boldsymbol{w} = (u,T)$ be a solution of Problem 1 and $0 < l < \infty$. If $G = 0$, then*

$$\eta_l(x) = \int_0^l (u(x,\psi) - \delta)^2\, d\psi \le \eta(0)\,\mathrm{e}^{-\nu x}; \tag{17}$$

$$\mu_l(x) = \int_0^l (T(x,\psi) - \varepsilon)^2\, d\psi \le \mu(0)\,\mathrm{e}^{-\lambda x}\,. \tag{18}$$

If $G > 0$ and $\lambda > \nu$ (the latter inequality corresponds to $\mathrm{Pr} < 1$*), then Eq.* (17) *should be substituted by the inequality*

$$\eta_l(x) = \int_0^l (u(x,\psi) - \delta)^2\, d\psi \le 2\eta(0)\Big(1 + \frac{4G^2\,\mathrm{e}^2}{\delta^2\varepsilon^2}\Big)\,\mathrm{e}^{-\nu x}\,. \tag{19}$$

Proof. Multiplying the first equation of system (5) by $u - \delta$, multiplying the second equation in (5) by $T - \varepsilon$, and integrating the resultant equations with respect to ψ over the interval $(0,l)$, we obtain the following energy relations:

$$\frac{d\eta_l(x)}{2dx} + \int_0^l a u_\psi^2\, d\psi = \int_0^l \frac{G}{uT}\,(T - \varepsilon)(u - \delta)\, d\psi; \tag{20}$$

$$\frac{d\mu_l(x)}{2dx} + \int_0^l \frac{a}{\mathrm{Pr}}\, T_\psi^2\, d\psi = 0. \tag{21}$$

First, we assume that $G = 0$. Then, using the inequalities

$$\eta_l(x) \le 4l^2 \int\limits_0^l u_\psi^2 \, d\psi, \qquad \mu_l(x) \le 4l^2 \int\limits_0^l T_\psi^2 \, d\psi, \tag{22}$$

we come to the differential inequalities

$$\frac{d\eta_l}{dx} + \nu\eta_l \le 0, \qquad \frac{d\mu_l}{dx} + \lambda\mu_l \le 0 \qquad \Big(\nu = \frac{a_0}{2l^2}, \quad \lambda = \frac{\nu}{\mathrm{Pr}}\Big).$$

Integrating these inequalities, we obtain estimates (17) and (18).

Now we assume that $G > 0$. Taking into account (18), we obtain from (20)

$$\frac{d\eta_l}{dx} + \nu\eta_l \le \gamma\sqrt{\eta_l}\,\sqrt{\mu_l} \le \gamma\sqrt{\eta_l}\,\sqrt{\mu_l(0)}\,\mathrm{e}^{-\lambda x/2} = \omega\sqrt{\eta}\,\mathrm{e}^{-\lambda x/2}, \tag{23}$$

where $\gamma = 2G/(\delta\varepsilon)$ and $\omega = \gamma\sqrt{\mu_l(0)}$. Integrating Eq. (23) under the condition $\lambda > \nu$, we come to the inequality

$$2\sqrt{\mu_l(x)\,\mathrm{e}^{\nu x}} - 2\sqrt{\mu_l(0)} \le \omega \int\limits_0^x \mathrm{e}^{-(\lambda-\nu)s/2}\, ds \le \omega\,\frac{2\,\mathrm{e}}{\lambda - \nu} = 2\beta < \infty,$$

which leads us to the desired estimate (19). The theorem is proved.

Remark 3. In Theorem 2.2, we have

$$\lambda = a_0/(2l^2\mathrm{Pr}\,) \to 0, \qquad \nu = a_0/(2l^2) \to 0 \qquad \text{as} \quad l \to \infty.$$

Now we derive some estimates independent of l by assuming additionally that $G = 0$ and the initial functions u_0 and T_0 are defined for $\psi \in [0,\infty]$ and

$$\int\limits_0^\infty |u_0 - \delta|\, d\psi \le C_u < \infty, \qquad \int\limits_0^\infty |T_0 - \varepsilon|\, d\psi \le C_T < \infty. \tag{24}$$

Next we use the well-known estimates (see, e.g., [2, 19, 20])

$$\int\limits_0^l |u - \delta|\, d\psi \le \int\limits_0^l |u_0 - \delta|\, d\psi \le C_u, \qquad \int\limits_0^l |T - \varepsilon|\, d\psi \le \int\limits_0^l |T_0 - \varepsilon|\, d\psi \le C_T \tag{25}$$

for solutions of system (5) with $i = G = 0$. Notice that (24) and (25) together with (10) imply that

$$\eta_\infty(x) = \int\limits_0^\infty |u_0 - \delta|^2\, d\psi \le C_u < \infty, \qquad \mu_\infty(x) = \int\limits_0^\infty |T_0 - \varepsilon|^2\, d\psi \le C_T < \infty.$$

Theorem 2.3. *Let $\boldsymbol{w} = (u, T)$ be a solution of Problem* 1 *with the initial data satisfying Eq.* (24) *and $G = 0$, then, we have*

$$\eta_l(x) = \int\limits_0^l (u - \delta)^2\, d\psi \le \frac{\eta_\infty(0)}{\sqrt{1 + 2x\nu\eta_\infty^2(0)}}, \qquad \nu = \frac{8a_0}{9C_u^4}; \tag{26}$$

$$\mu_l(x) = \int\limits_0^l (T - \varepsilon)^2\, d\psi \le \frac{\mu_\infty(0)}{\sqrt{1 + 2x\lambda\mu_\infty^2(0)}}, \qquad \lambda = \frac{8a_0}{9C_T^4\mathrm{Pr}}. \tag{27}$$

Proof. To prove the theorem, we use again the energy relations (20) and (21) with $G = 0$. Instead of (22), however, we apply the interpolation inequality

$$\eta_l^3(x) = \Big(\int\limits_0^l (u - \delta)^2\, d\psi\Big)^3 \le \Big(\frac{3}{2}\Big)^2 \Big(\int\limits_0^l |u - \delta|\, d\psi\Big)^4 \int\limits_0^l |u_\psi|^2\, d\psi.$$

Then, Eqs. (20) and (25) lead us to the differential inequality

$$\frac{d\eta_l}{dx} + \nu\eta_l^3 \le 0, \qquad \nu = \frac{8a_0}{9C_u^4}.$$

Integrating the last inequality, we obtain

$$\eta_l(x) = \int_0^l (u-\delta)^2\, d\psi \le \frac{\eta(0)}{\sqrt{1+2x\nu\eta^2(0)}} \le \frac{\eta_\infty(0)}{\sqrt{1+2x\nu\eta_\infty^2(0)}}.$$

Passing to the limit as $l \to \infty$, we complete the proof of estimate (26). The proof of estimate (27) is the same. Theorem 2.3 is proved.

REMARK 4. Assume that $G = 0$ and $\boldsymbol{w}$ is the solution of Problem 1. Then,

$$\int_0^\infty |\boldsymbol{w}(x,\psi) - \boldsymbol{w}_{\delta,\varepsilon}|^4\, dx \to 0 \quad \text{as} \quad \psi \to \infty, \tag{28}$$

where $\boldsymbol{w} = (u,T)$ and $\boldsymbol{w}_{\delta,\varepsilon} = (\delta,\varepsilon)$. It follows from Eqs. (20) and (21) that

$$\sup_{0\le x\le\infty} \int_0^\infty |\boldsymbol{w} - \boldsymbol{w}_{\delta,\varepsilon}|^2\, d\psi + \int_0^\infty\int_0^\infty |\boldsymbol{w}_\psi|^2\, d\psi\, dx \le C_0(\varepsilon,\delta) \int_0^\infty |\boldsymbol{w}_0 - \boldsymbol{w}_{\delta,\varepsilon}|^2\, d\psi.$$

Using the multiplicative inequality

$$\int_0^\infty |\boldsymbol{w}(x,\psi) - \boldsymbol{w}_{\delta,\varepsilon}|^4\, dx \le 4\Big(\sup_{0\le x\le\infty} \int_\psi^\infty |\boldsymbol{w}(x,\psi) - \boldsymbol{w}_{\delta,\varepsilon}|^2\, d\psi\Big) \int_0^\infty\int_\psi^\infty |\boldsymbol{w}_\psi|^2\, d\psi\, dx$$

and the Lebesgue theorem, we obtain (28).

2.2. *Existence and Uniqueness of Local Solutions of Problem* 2. First we introduce new variables $x = x$ and $\eta = \psi/f(x)$. Using the formulas

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x} - \frac{\eta f'(x)}{f}\,\frac{\partial}{\partial\eta}, \qquad \frac{\partial}{\partial\psi} = \frac{1}{f}\,\frac{\partial}{\partial\eta}, \qquad \frac{D(x,\eta)}{D(x,\psi)} = \frac{1}{f(x)},$$

we reduce Problem 2 to the problem

$$\begin{aligned} \frac{\partial u}{\partial x} - \frac{\eta f'(x)}{f(x)}\,\frac{\partial u}{\partial\eta} &= \frac{1}{f^2(x)}\,\frac{\partial}{\partial\eta}\Big(a\,\frac{\partial u}{\partial\eta}\Big) + c,\\ \frac{\partial T}{\partial x} - \frac{\eta f'(x)}{f(x)}\,\frac{\partial T}{\partial\eta} &= \frac{1}{f^2(x)}\,\frac{1}{\mathrm{Pr}}\,\frac{\partial}{\partial\eta}\Big(a\,\frac{\partial u}{\partial\eta}\Big); \end{aligned} \tag{29}$$

$$\begin{gathered} \frac{\partial u}{\partial\eta} = \frac{\partial T}{\partial\eta} = 0, \quad \eta = 0, \quad x > 0, \qquad u = \delta, \quad T = \varepsilon, \quad \eta = 1, \quad X > 0,\\ u = u_0(\eta f(0)), \quad T = T_0(\eta f(0)) \qquad \text{for} \quad x = 0, \quad 0 \le \eta \le 1 \end{gathered} \tag{30}$$

in a semistrip $\Omega = \{(x,\eta) \in \mathbb{R}^2\colon\ 0 < x < \infty,\ 0 < \eta < 1\}$. Here

$$f(0) = \int_0^l \frac{u_0(s)}{T_0(s)}\, ds > 0$$

is a given constant, and the unknown function $f(x)$ is defined by means of a nonlocal operator over u and T and their derivatives with respect to η by the relation

$$\begin{gathered} f(x)f'(x) = \Xi(u,T,u_\eta,T_\eta),\\ \Xi =: \Big(T^{\sigma-1}u_\eta - \frac{uT^{\sigma-2}T_\eta}{\mathrm{Pr}}\Big)\Big|_{\eta=1} + \frac{u(x,1)}{T(x,1)} \int_0^1 \Big(-T^{\sigma-1}T_\eta u_\eta\Big(\frac{1}{u} + \frac{1}{\mathrm{Pr}}\Big) + \frac{2}{u^2}\,T^{\sigma-1}u_\eta^2\Big)\, ds. \end{gathered} \tag{31}$$

To prove the existence theorem, we derive some *a priori* estimates for the solution in addition to (10) and (11). Using the definition of the stream function, we obtain

$$\psi(x,r)=\int_0^r \rho(x,r)u(x,r)\,dr,$$
$$\frac{l}{C_0}\le l\min(\rho u)\le \psi(x,l)=f(x)\le l\max(\rho u)\le C_0 l. \tag{32}$$

Multiplying the first equation of system (29) by $f(x)(u-\delta)$, multiplying the second equation of system (29) by $f(x)(T-\varepsilon)$, and integrating the resultant equations with respect to η and x, we come to the relations

$$f(x)\int_0^1 (u(x,\eta)-\delta)^2\,d\eta+2\int_0^x\int_0^1 \frac{a}{f(x)}\Big(\frac{\partial u}{\partial\eta}\Big)^2\,d\eta\,dx=f(x)\int_0^1(u_0(\eta f(0))-\delta)^2\,d\eta+\int_0^x fI_c\,dx,$$

$$f(x)\int_0^1 (T(x,\eta)-\varepsilon)^2\,d\eta+2\int_0^x\int_0^1 \frac{a}{f(x)\mathrm{Pr}}\Big(\frac{\partial T}{\partial\eta}\Big)^2\,d\eta\,dx=f(x)\int_0^1(T_0(\eta f(0))-\varepsilon)^2\,d\eta,$$

where

$$I_c=\int_0^1\frac{G}{uT}\,(u-\delta)(T-\varepsilon)\,d\eta.$$

According to Eqs. (10), (11), and (32), these relations give us the estimate

$$\int_0^x\int_0^1 |\boldsymbol{w}_\eta|^2\,d\eta\,dx\le C(l,\varepsilon,\delta).$$

Multiplying the first equation of system (29) by $u_{\eta\eta}$ and the second equation by $T_{\eta\eta}$, we obtain

$$\frac{1}{2}\frac{d}{dx}\int_0^1 u_\eta^2\,d\eta+\frac{1}{f^2(x)}\int_0^1 au_{\eta\eta}^2\,d\eta=I+I_1+\Lambda_1,$$
$$\frac{1}{2}\frac{d}{dx}\int_0^1 T_\eta^2\,d\eta+\frac{1}{f^2(x)\mathrm{Pr}}\int_0^1 aT_{\eta\eta}^2\,d\eta=I_2+\Lambda_2, \tag{33}$$

where

$$I=\int_0^1 cu_{\eta\eta}\,d\eta,\qquad I_1=-\frac{f'(x)}{f(x)}\int_0^1\eta u_\eta u_{\eta\eta}\,d\eta,$$

$$\Lambda_1=\frac{1}{f^2(x)}\int_0^1(a_u u_\eta^2+a_T u_\eta T_\eta)T_{\eta\eta}\,d\eta,$$

$$I_2=-\frac{f'(x)}{f(x)}\int_0^1\eta T_\eta T_{\eta\eta}\,d\eta,\qquad \Lambda_2=\frac{1}{f^2(x)\mathrm{Pr}}\int_0^1(a_T T_\eta^2+a_u u_\eta T_\eta)u_{\eta\eta}\,d\eta.$$

It is easy to verify that

$$|I|\le\epsilon\int_0^1 u_{\eta\eta}^2\,d\eta+C\int_0^1|\boldsymbol{w}|^2\,d\eta,$$

$$(|\Lambda_1|,|\Lambda_2|) \le C\int_0^1 |\boldsymbol{w}_\eta|^2|\boldsymbol{w}_{\eta\eta}|\,d\eta \le \epsilon\int_0^1 |\boldsymbol{w}_{\eta\eta}|^2\,d\eta + C\int_0^1 |\boldsymbol{w}_\eta|^4\,d\eta, \quad \epsilon\in(0,1) \tag{34}$$

with some $C = C(\epsilon,\varepsilon,\delta,l)$. Using Eqs. (31), we obtain the estimates

$$(|I_1|,|I_2|) \le \epsilon\int_0^1 |\boldsymbol{w}_{\eta\eta}|^2\,d\eta + Cf'^2(x)\int_0^1 |\boldsymbol{w}_\eta|^2\,d\eta, \qquad \epsilon\in(0,1),$$

where

$$f'^2(x)\int_0^1 |\boldsymbol{w}_\eta|^2 d\eta \le C\Big[|\boldsymbol{w}_\eta(x,1)|^2 + \int_0^1 |\boldsymbol{w}_\eta|^4\,d\eta\Big]\int_0^1 |\boldsymbol{w}_\eta|^2\,d\eta.$$

Next we introduce the function

$$Y(x) = \int_0^1 |\boldsymbol{w}_\eta|^2\,d\eta$$

and use the multiplicative inequalities

$$|\boldsymbol{w}_\eta(x,\eta)|^2 \le 2Y^{1/2}(x)\Big(\int_0^1 |\boldsymbol{w}_{\eta\eta}|^2\,d\eta\Big)^{1/2}, \qquad \int_0^1 |\boldsymbol{w}|^2\,d\eta \le 4Y(x), \tag{35}$$

$$\int_0^1 |\boldsymbol{w}_\eta|^4\,d\eta \le 2Y^{3/2}(x)\Big(\int_0^1 |\boldsymbol{w}_{\eta\eta}|^2\,d\eta\Big)^{1/2}$$

to obtain

$$f'^2(x)\int_0^1 |\boldsymbol{w}_\eta|^2\,d\eta \le \epsilon\int_0^1 |\boldsymbol{w}_{\eta\eta}|^2\,d\eta + CY^5. \tag{36}$$

Joining Eqs. (33)–(36) and choosing a suitable $\epsilon > 0$, we obtain the inequality

$$\frac{dY}{dx} + \int_0^1 |\boldsymbol{w}_{\eta\eta}|^2\,d\eta \le CY^5.$$

From here it follows that

$$Y(x) \le \Big(\frac{Y^4(0)}{1 - xCY^4(0)}\Big)^{1/4} < \infty \qquad \text{if} \quad 0 \le x \le X_0 < X^* = \frac{1}{CY^4(0)} \tag{37}$$

and, consequently,

$$Y(x) + \int_0^x\int_0^1 |\boldsymbol{w}_{\eta\eta}|^2\,d\eta\,dx \le C(Y(0),X_0) \qquad \text{if} \quad x \le X_0 < X^*. \tag{38}$$

The last estimate is valid for any given X^* if $Y(0) = \int_0^1 |\boldsymbol{w}_\eta(0,\eta)|^2\,d\eta$ satisfies the inequality $X^*CY^4(0) < 1$. Estimate (38) guarantees a stronger smoothness of the solution. In fact, using Eqs. (31) and (35), we find that

$$|f'(x)|^4 \le C\Big(|\boldsymbol{w}_\eta(x,1)|^4 + Y^4\Big) \le C\Big(Y\int_0^1 \boldsymbol{w}_{\eta\eta}^2\,d\eta + Y^4\Big) \in L^1(0,X_0).$$

Hence, we have

$$\|f'(x)\|_{L^4(0,X_0)} \le C(l,\delta,\varepsilon,X_0). \tag{39}$$

Now we can consider equations of system (29) as independent equations with given coefficients $f'(x)$, $f(x)$, a, and b and apply the theory of parabolic equations [18].

To prove the existence theorem, we construct a completely continuous operator to apply Schauder's fixed point theorem.

Using (31), we define a one-parameter set of operators

$$\Phi_\lambda = \Phi_\lambda(\boldsymbol{w},\boldsymbol{w}_\eta) = \frac{1}{2}\,\lambda\Xi(\boldsymbol{w},\boldsymbol{w}_\eta|x) \Big/ \Big(2\lambda\int\limits_0^1 \Xi(\boldsymbol{w},\boldsymbol{w}_\eta|s)\,ds + f^2(0)\Big), \qquad \lambda\in[0,1].$$

Next we consider the system of equations

$$\begin{gathered}\frac{\partial u}{\partial x} - \eta g'(x)\,\frac{\partial u}{\partial \eta} = \frac{\partial}{\partial \eta}\Big(\mathrm{e}^{2g(x)}\,a\,\frac{\partial u}{\partial \eta}\Big),\\ \frac{\partial T}{\partial x} - \eta g'(x)\,\frac{\partial T}{\partial \eta} = \frac{1}{\Pr}\,\frac{\partial}{\partial \eta}\Big(\mathrm{e}^{2g(x)}\,b\,\frac{\partial u}{\partial \eta}\Big);\end{gathered} \tag{40}$$

$$g'(x) = \Phi_\lambda(\boldsymbol{w},\boldsymbol{w}_\eta), \qquad g(0)=\ln f(0) \tag{41}$$

under the boundary and initial conditions (30). Note that this problem with $\lambda = 0$ was considered in Sec. 2.1. The above-obtained *a priori* estimates remain valid for any $\lambda\in[0,1]$. Taking into account Eqs. (32) and (39), we define a family of the functions $\varphi\in M$ such that

$$\varphi(0)=g(0), \qquad \ln(l/C_0)\le\varphi(x)\le\ln lC_0, \qquad \varphi'\in L^2.$$

Let $\varphi\in M$ be an arbitrary given function. We substitute φ to (40) instead of g. Then, the corresponding problem has at least one solution $\boldsymbol{w}=(u,T)$, which defines a nonlinear operator $\boldsymbol{\Pi}_\lambda\colon\ \varphi\to\boldsymbol{w}$. Later, using (41), we can define the operator

$$F_\lambda\colon\ \varphi\to g = \int\limits_0^x \Phi_\lambda(\boldsymbol{w},\boldsymbol{w}_\eta)\,ds + g(0).$$

According to the above-mentioned *a priori* estimates and the theory of parabolic equations, the operator F_λ is completely continuous and satisfies all conditions of Schauder's theorem. Therefore, the equation $\varphi=F_\lambda(\varphi)$ has at least one solution, which defines the functions $\boldsymbol{w}=(u,T)$. Thus, we prove the following result.

Theorem 2.4. *Let the functions* $(u_0,T_0)\in C^{2+\alpha}[0,1]$ $(0<\alpha<1)$ *satisfy the corresponding compatibility conditions. Then, Problem* 2 *has a unique classical solution* $\boldsymbol{w}=(u(x,\eta),T(x,\eta))$ *on the interval* $x\in[0,X_0]\subset[0,X^*)$, *where* $X^*=X^*(Y(0))>0$ *is defined by* (37). *Moreover, the solution exists for any finite value of* X_0, *provided that* $Y(0)=\int\limits_0^1(u_{0\eta}'^2+T_{0\eta}'^2)\,d\eta$ *is sufficiently small.*

Now we start to prove the uniqueness of the classical solutions.

Theorem 2.5. *The classical solution of Problem* 2 *is unique.*

Proof. Let $\boldsymbol{w}_i=(u_i,T_i)$ $(i=1,2)$ be two different classical solutions and $\boldsymbol{w}=\boldsymbol{w}_1-\boldsymbol{w}_2$. Then, the vector function $\boldsymbol{w}$ is the solution of the problem

$$\boldsymbol{w}_x = A\boldsymbol{w}_{\eta\eta} + B\boldsymbol{w}_\eta + C\boldsymbol{w} + D\boldsymbol{w}_\eta(x,1) + \int\limits_0^1 E\boldsymbol{w}_\eta\,d\eta; \tag{42}$$

$$\frac{\partial \boldsymbol{w}}{\partial \eta}=0,\quad \eta=0,\quad x>0, \qquad\qquad \boldsymbol{w}=0,\quad \eta=1,\quad x>0,$$

$$\boldsymbol{w}=0, \qquad x=0, \quad 0\le\eta\le1$$

with bounded matrices

$$|A, B, C, D, E| \leq C < \infty \tag{43}$$

and a positive matrix A

$$a_0(\boldsymbol{\xi}, \boldsymbol{\xi}) \leq (A\boldsymbol{\xi}, \boldsymbol{\xi}) \qquad \forall \boldsymbol{\xi} \in \mathbb{R}^2. \tag{44}$$

Multiplying Eq. (42) by $\boldsymbol{w}_{\eta\eta}$ and integrating, we come to the integral relation

$$\frac{1}{2}\frac{d}{dx}\int_0^1 |\boldsymbol{w}_\eta|^2\, d\eta + \int_0^1 (A\boldsymbol{w}_{\eta\eta}, \boldsymbol{w}_{\eta\eta})\, d\eta = I, \tag{45}$$

with

$$I = -\int_0^1 \Big(B\boldsymbol{w}_\eta + C\boldsymbol{w} + D\boldsymbol{w}_\eta(x,1) + \int_0^1 E\boldsymbol{w}_\eta\, d\eta\Big)\boldsymbol{w}_{\eta\eta}\, d\eta.$$

Taking into account Eqs. (35), (43), and (44), we can evaluate I in the following way:

$$|I| \leq \frac{1}{2}\int_0^1 (A\boldsymbol{w}_{\eta\eta}, \boldsymbol{w}_{\eta\eta})\, d\eta + C\int_0^1 |\boldsymbol{w}_\eta|^2 d\eta.$$

Joining (45) and the last estimate, we obtain the differential inequality

$$\frac{d}{dx}\int_0^1 |\boldsymbol{w}_\eta|^2 d\eta \leq C\int_0^1 |\boldsymbol{w}_\eta|^2 d\eta,$$

which completes the proof of the theorem.

REMARK 5. The constructed solution defines the homeomorphism between the domain $\Omega_{x,r} = \{(x,\psi) \in \mathbb{R}^2\colon 0 < x < X^*,\ 0 < r < l\}$ in the physical variables and the domain $\Omega_{X,\psi} = \{(x,\psi) \in \mathbb{R}^2\colon\ 0 < x < X^*,\ 0 < \psi < f(x,l)\}$. This classical solution $\boldsymbol{w} = (u, T)$ determines the classical solution $[v(x,r), u(x,r), T(x,r)]$ of system (1) satisfying conditions (2) (see Remark 1). The second component of the velocity vector v is defined by Eq. (16).

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